JOURNAL OF DIFFERENTIAL EQUATIONS 22, 442–452 (1976)

Shock Waves in the Nonisentropic Gas Flow

TAI-PING LIU

Department of Mathematics, University of Maryland, College Park, Maryland 20742

Received February 3, 1975; revised July 1, 1975

In this paper we propose an extended entropy condition for general systems of hyperbolic conservation laws with several space variables. This entropy condition generalizes the well-known condition (E) of Volpert for a single conservation law with several space variables and reduces to the entropy condition proposed earlier by the author for systems with one space variable. The Riemann problem for general nonisentropic gas equations has a unique solution for initial data with arbitrarily large jumps. The occurrence of a vacuum region is observed. The projections of shock curves on the pressure–velocity plane are analyzed so as to study the interaction of weak shocks. Our results differ markedly from those of previous works in that we do not assume the equation of state to be polytropic. In fact our assumptions on the equation of state allow the pressure to be a nonconvex function of specific volume.

The Riemann problem for this general system of gas equations was also treated by B. Wendroff when the initial data are near constant.

1. General Conservation Laws

We consider the general system of conservation laws

$$U_t + \sum_{i=1}^{n} F^i(U)_{x_i} = 0, \qquad -\infty < x_i < \infty,\ t \geqslant 0, \tag{1.1}$$

where $U = (u_1, ..., u_m)^t$, $F^i = (f_1{}^i, ..., f_m{}^i)^t$, $U = U(x, t)$ and $x = (x_1, ..., x_n)$.

Assume that (1.1) is *hyperbolic*, that is, for any nonzero vector $\nu = (\nu_1, ..., \nu_n)$, the matrix $M(\nu) \equiv \sum_{i=1}^{n} \nu_i(\partial F^i/\partial U)$ has real eigenvalues $\lambda_\nu{}^1 \leqslant \lambda_\nu{}^2 \leqslant \cdots \leqslant \lambda_\nu{}^m$ with corresponding linearly independent eigenvectors $r_\nu{}^1, r_\nu{}^2, ..., r_\nu{}^m$ for each U. We sometimes simply write $\lambda_\nu{}^i$ and $r_\nu{}^i$ as λ^i and r^i.

A bounded measurable function $U(x, t)$ is a *weak* solution of (1.1) with initial data

$$U(x, 0) = U_0(x) \tag{1.2}$$

if for any $\Phi(x, t) \in C_0^\infty(\mathbb{R}^n; \mathbb{R}^+)$,

$$\int_{\substack{x\in\mathbb{R}^n\\ t>0}} U(x, t)\, \Phi_t(x, t) + \sum_{i=1}^{n} F^i((x, t))\Phi_{x_i}(x, t)\, dx\, dt + \int_{x\in\mathbb{R}^n} U_0(x)\, \Phi(x, 0)\, dx = 0, \tag{1.3}$$

where 0 is the zero vector.

Suppose that a weak solution $U(x, t)$ of (1.1) and (1.2) has a discontinuity of the first kind at (x, t) along the surface $g(x, t) = 0$ which has a nonzero normal $(\nu_0, \nu_1, \ldots, \nu_n)$, $\nu_0 = \partial g/\partial t$, $\nu_1 = \partial g/\partial x_1, \ldots, \nu_n = \partial g/\partial x_n$. Then the following *Hugoniot* condition holds.

$$\sum_{i=1}^{n} \nu_i(F^i(U_+) - F^i(U_-)) = \nu_0(U_+ - U_-), \tag{H}$$

where $U_+ = \lim_{\delta\to 0_+} U(x + \delta\nu, t)$ and $U_- = \lim_{\delta\to 0_-} U(x + \delta\nu, t)$. It is easily seen that $\nu \equiv (\nu_1, \ldots, \nu_n)$ is nonzero.

For fixed $\nu = (\nu_1, \ldots, \nu_n)$ and $U^0 = (u_1^0, \ldots, u_m^0)$, the *shock curves* is defined as:

$$S_\nu(U_0) = \left\{ U \,\middle|\, \frac{\sum_{i=1}^n \nu_i(f_j^i(U) - f_j^i(U_0))}{u_j - u_j^0} = \frac{\sum_{i=1}^n \nu_i(f_k^i(U) - f_k^i(U_0))}{u_k - u_k^0} \equiv \sigma_\nu(U, U_0), \quad j, k = 1, 2, \ldots, m \right\}$$

where $\sigma_\nu(U, U_0)$ is the *shock speed.* Condition (H) says that $U_+ \in S_\nu(U_-)$ and $\nu_0 = \sigma_\nu(U_-, U_+)$. Motivated by Lax [3], we assume that $S_\nu(U_0)$ consists of n smooth curves $S_\nu^j(U_0)$, $j = 1, 2, \ldots, m$, in the U-plane, such that for any $U \in S_\nu^j(U_0)$, $\lambda_\nu^{j-1}(U) \leqslant \sigma_\nu(U, U_0) \leqslant \lambda_\nu^{j+1}(U)$ (cf. [4]). For $U_1 \in S_\nu^j(U_0)$, (U_0, U_1) is *admissible* if it satisfies the following *extended entropy condition* (E).

$$\sigma_\nu(U_0, U) \geqslant \sigma_\nu(U_0, U) \tag{E}$$

for any $U \in S_\nu^j(U_0)$ between U_1 and U_0. If a discontinuity (U_-, U_+) is admissible, then, under general assumptions (cf. [4]), it can be shown that (U_-, U_+) satisfies the following generalized *Lax's shock inequalities* (L) (cf. [3]).

$$\lambda_\nu^j(U_-) \geqslant \sigma_\nu(U_-, U_+) \geqslant \lambda_\nu^j(U_+) \tag{L}$$

when $U_+ \in S_\nu^j(U_-)$. Condition (E) is equivalent to (L) if $(\nabla\lambda_\nu^j) \cdot r_\nu^j \neq 0$, that is, (1.1) is *genuinely nonlinear* with respect to ν. Condition (E) reduces to the well-known extended entropy condition for a single conservation law with several space variables (cf. [6]).

The Riemann problem (1.1) is the Cauchy problem with initial data

$$\begin{aligned} U(x, 0) &= U_l \qquad \text{for} \quad x \cdot \nu < 0, \\ &= U_r \qquad \text{for} \quad x \cdot \nu > 0, \end{aligned} \tag{1.4}$$

where U_l, U_r, and ν are constants. Similar to the one space variable case, we can treat the Riemann problem and relate a shock satisfying condition (E) to the traveling wave solution of the associated viscosity equations.

2. Shock Curves for Nonisentropic Gas Equations

The nonisentropic gas equations in Eulerian coordinates are (see, e.g., [2])

$$\begin{aligned} &\rho_t + (\rho u)_x + (\rho v)_y + (\rho w)_z = 0 \\ &\qquad\qquad\qquad\qquad \text{(conservation of mass)}, \\ &(\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y + (\rho uw)_z = 0 \\ &\qquad\qquad\qquad\qquad \text{(conservation of momentum)}, \\ &(\rho v)_t + (\rho vu)_x + (\rho v^2 + p)_y + (\rho vw)_z = 0, \\ &(\rho w)_t + (\rho wu)_x + (\rho wv)_y + (\rho w^2 + p)_z = 0, \\ &(\rho E)_t + (\rho uE + pu)_x + (\rho vE + pv)_y + (\rho wE + pw)_z = 0 \\ &\qquad\qquad\qquad\qquad \text{(conservation of energy)}, \\ &p = p(\tau, e) = p(\tau, s) \geqslant 0 \qquad \text{(equation of state)}, \end{aligned} \tag{2.1}$$

where $t \geqslant 0$, $(x, y, z) \in \mathbb{R}^3$, and (u, v, w), ρ $(\geqslant 0)$, p, e, s are the velocity, density, pressure, internal energy, and entropy, and $E = \frac{1}{2}(u^2 + v^2 + w^2) + e$, the total energy, $\tau = 1/\rho$ the specific volume. We assume that

$$p_e \equiv \partial p(\tau, e)/\partial e > 0 \qquad \text{and} \qquad \tilde{p}_\tau = \partial p(\tau, e)/\partial \tau < 0. \tag{2.2}$$

By the thermodynamics identity $de = -p\, d\tau + T\, ds$, we have $p_\tau \equiv \partial p(\tau, s)/\partial \tau = -pp_e + \tilde{p}_\tau < 0$ and $p_s \equiv \partial p(\tau, s)/\partial s = Tp_e > 0$.

Since the system (2.1) is invariant under rotation, without loss of generality, we will only observe discontinuities propagating along x-axis. That is, we take the vector ν in Section 1 to be $(1, 0, 0)$, and write S_ν and $M(\nu)$ as S and M. Then

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -u^2 + p^1 & 2u + p^2 & p^3 & p^0 & p^0 \\ -uv & v & u & 0 & 0 \\ -uw & w & 0 & u & 0 \\ -uE - up^1 - (up/\rho) & E + up^2 + (p/\rho) & up^3 & up^0 & u + up^0 \end{bmatrix}.$$

Here the vector U in Section 1 is taken to be $(\rho, \rho u, \rho v, \rho w, \rho E)$ and (p^i) is the gradient of p with respect to U. Their values are

$$p^1 = \frac{-\tilde{p}_\tau - p_e E}{\rho^2} + \frac{u^2 + v^2 + w^2}{\rho} p_e, \qquad p^2 = -\frac{u p_e}{\rho},$$

$$p^3 = -\frac{v p_e}{\rho}, \qquad p^4 = -\frac{w p_e}{\rho}. \qquad p^0 = \frac{p_e}{\rho}.$$

The eigenvalues and eigenvector of M are

$$\begin{aligned}
&\lambda_1 = u - \frac{(-p_\tau)^{1/2}}{\rho}, \qquad \lambda_2 = u, \qquad \lambda_3 = u + \frac{(-p_\tau)^{1/2}}{\rho},\\
&r_1 = \left(1, u - \frac{(-p_\tau)^{1/2}}{\rho}, v, w, \frac{\rho E + p - u(-p_\tau)^{1/2}}{\rho}\right),\\
&r_3 = \left(1, u + \frac{(-p_\tau)^{1/2}}{\rho}, v, w, \frac{\rho E + p + u(-p_\tau)^{1/2}}{\rho}\right).
\end{aligned} \tag{2.3}$$

Here u is the multiple eigenvalues and a vector r is a eigenvector corresponding to $\lambda_2 = u$ if and only if $r \cdot \nabla u = r \cdot \nabla p = 0$. Assumption (2.2) implies that λ_1 and λ_3 are real and (2.1) is hyperbolic. In what follows, we denote by $\nabla \equiv \partial/\partial U$. We have

$$\begin{aligned}
&r_1 \cdot \nabla u = \frac{-(-p_\tau)^{1/2}}{\rho^2}, \qquad r_3 \cdot \nabla u = \frac{(-p_\tau)^{1/2}}{\rho^2},\\
&r_1 \cdot \nabla p = r_3 \cdot \nabla p = -p_\tau/\rho^2,\\
&r_1 \cdot \nabla\lambda_1 = -\frac{p_{\tau\tau}}{2(-p_\tau\rho^3)^{1/2}}, \qquad r_3 \cdot \nabla\lambda_3 = \frac{p_{\tau\tau}}{2(-p_\tau\rho^3)^{1/2}},\\
&r_1 \cdot \nabla s = r_3 \cdot \nabla s = 0.
\end{aligned} \tag{2.4}$$

The Hugoniot condition for U and U_0 are

$$\begin{aligned}
\sigma &= \frac{\rho u - \rho_0 u_0}{\rho - \rho_0} = \frac{(\rho u^2 + p) - (p_0 u_0{}^2 + p_0)}{\rho u - \rho_0 u_0} = \frac{\rho u v - \rho_0 u_0 v_0}{\rho v - \rho_0 v_0}\\
&= \frac{\rho u w - \rho_0 u_0 w_0}{\rho w - \rho_0 w} = \frac{(E\rho u + pu) - (E_0\rho_0 u_0 + p_0 u_0)}{\rho E - \rho_0 E_0}.
\end{aligned} \tag{2.5}$$

From (2.5) it follows that $\sigma = u_0$ if and only if $u = u_0$ if and only if $p = p_0$. In other words, U and U_0 are connected by a *contact discontinuity* with speed $\sigma = u_0 = u$ if and only if u and p are unchanged across the discontinuity. Similarly, if $\sigma \neq u_0$, then $u \neq u_0$, $p \neq p_0$ and $v = v_0$, $w = w_0$. That is, the velocity component tangent to the surface of the discontinuity suffers no change across a forward or backward shock wave; while across a contact discontinuity, the velocity component normal to the surface of the discontinuity remains unchanged. The *forward shock curve* through U_0 is the set $S_3(U_0) \equiv \{U \in S(U_0) \mid \sigma(U, U_0) > u_0\}$ and the *backward shock curve* through U_0 is $S_1(U_0) \equiv \{U \in S(U_0) \mid \sigma(U, U_0) < u_0\}$. The following lemma shows that $S_1(U_0)$ and $S_3(U_0)$ are smooth curves and are functions of both u and p.

LEMMA 2.1. *Given $\tilde{u} \neq u_0$, there exists at most one point on $\{U \mid u = \tilde{u}\} \cap S_i(U_0)$, $i = 1$ or 3. Similarly, given $\tilde{p} \neq p_0$, there exists at most one point on $\{U \mid p = \tilde{p}\} \cap S_i(U_0)$, $i = 1$ or 3.*

Proof. Suppose that two points $\bar{U} \neq \bar{\bar{U}}$, $\bar{U} = \{\bar{\rho}, \bar{\rho}\bar{u}, \bar{\rho}v_0, \bar{\rho}w_0, \bar{\rho}\bar{E}\}$, $\bar{\bar{U}} = \{\bar{\bar{\rho}}, \bar{\bar{\rho}}\bar{\bar{u}}, \bar{\bar{\rho}}v_0, \bar{\bar{\rho}}w_0, \bar{\bar{\rho}}\bar{\bar{E}}\}$, are on $S(U_0)$ with shock speeds $\bar{\sigma} = \sigma(\bar{U}, U_0) \neq u_0$ and $\bar{\bar{\sigma}} = \sigma(\bar{\bar{U}}, U_0) \neq u_0$. To prove the first part of the lemma, we let $\bar{u} = \bar{\bar{u}} = u$ and show that

$$(\bar{\sigma} - u_0)(\bar{\bar{\sigma}} - u_0) < 0. \tag{2.6}$$

From (2.5), it follows that for any $U \in S(U_0)$,

$$\begin{aligned} \rho - \rho_0 &= \frac{\rho_0(u - u_0)}{\sigma - u} = \frac{\rho(u - u_0)}{\sigma - u_0}, \\ p - p_0 &= \rho_0(u - u_0)(\sigma - u_0), \\ e - e_0 &= \tfrac{1}{2}(u - u_0)^2 + \frac{p_0(u - u_0)}{\rho_0(\sigma - u_0)}. \end{aligned} \tag{2.7}$$

Suppose that (2.6) is false, that is, $(\bar{\sigma} - u_0)(\bar{\bar{\sigma}} - u_0) \geqslant 0$. Without loss of generality, assume that $\bar{\sigma} \geqslant \bar{\bar{\sigma}}$. When $u > u_0$, we have, from (2.7), $\bar{\rho} \leqslant \bar{\bar{\rho}}$, $\bar{p} \geqslant \bar{\bar{p}}$ and $\bar{e} \leqslant \bar{\bar{e}}$. On the other hand, since $\bar{p} - \bar{\bar{p}} = (\bar{\rho} - \bar{\bar{\rho}})p_\rho(\tilde{\rho}, \tilde{e}) + (\bar{e} - \bar{\bar{e}})p_e(\tilde{\rho}, \tilde{e})$ for some $(\tilde{\rho}, \tilde{e})$ between $(\bar{\rho}, \bar{e})$ and $(\bar{\bar{\rho}}, \bar{\bar{e}})$, (2.2) implies that if $\bar{\rho} \leqslant \bar{\bar{\rho}}$ and $\bar{e} \leqslant \bar{\bar{e}}$, then $\bar{p} \leqslant \bar{\bar{p}}$. Therefore we have $\bar{\rho} = \bar{\bar{\rho}}$, $\bar{p} = \bar{\bar{p}}$ and $\bar{e} = \bar{\bar{e}}$ and so $\bar{U} = \bar{\bar{U}}$ which is a contradiction. The case $u < u_0$ is treated similarly. This proves the first part of the lemma.

Next, assume that $\bar{p} = \bar{\bar{p}} = p$. We only prove the second half of the lemma when $p \geqslant p_0$. From (2.5), we have for $U \in S(U_0)$,

$$\begin{aligned} 1 - \frac{\rho_0}{\rho} &= \frac{p - p_0}{\rho_0(\sigma - u_0)^2}, \\ e - e_0 &= \frac{1}{2}\frac{p^2 - p_0^2}{\rho_0^2(\sigma - u_0)^2}. \end{aligned} \tag{2.8}$$

We first prove that $(\bar{\sigma} - u_0)^2 = (\bar{\bar{\sigma}} - u_0)^2$. Suppose otherwise. $(\bar{\sigma} - u_0)^2 > (\bar{\bar{\sigma}} - u_0)^2$. Then, by (2.8), $\bar{\rho} < \bar{\bar{\rho}}$ and $\bar{e} < \bar{\bar{e}}$ and so, by (2.5), $\bar{\bar{p}} - \bar{p} > 0$ which is a contradiction. Similarly, it is impossible to have $(\bar{\sigma} - u_0)^2 < (\bar{\bar{\sigma}} - u_0)^2$. To prove (2.6), it remains to show that $\bar{\sigma} - u_0 \neq \bar{\bar{\sigma}} - u_0$. Indeed, if $\bar{\sigma} - u_0 = \bar{\bar{\sigma}} - u_0$, then $\bar{p} = \bar{\bar{p}}$ and $\bar{e} = \bar{\bar{e}}$ by (2.8), and so $\bar{u} = \bar{\bar{u}}$ by (2.7). This implies that $\bar{U} = \bar{\bar{U}}$ which is a contradiction. The proof of the lemma is complete. Q.E.D.

3. The Riemann Problem

The Riemann problem for nonisentropic gas equations with one space variable was treated in [4]. It was proved that the solution ot the Riemann problem is unique. In this section we give a simpler proof to the uniqueness theorem. We also observe the existence of the solution to the Riemann problem.

From the investigation of shock curves of (2.1) in Section 2, we find that the shock curves and rarefaction curves for (2.1) are analogous to that for equations with one space variable:

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2 + p)_x &= 0, \\ (\rho E)_t + (\rho u E + p u)_x &= 0. \end{aligned} \tag{3.1}$$

Therefore, for simplicity we consider the Riemann problem (3.1) with initial data

$$\begin{aligned} U &= U_l \qquad \text{for} \quad x < 0, \\ U &= U_r \qquad \text{for} \quad x > 0. \end{aligned} \tag{3.2}$$

where $U = (\rho, \rho u, \rho E)$, $U_l = (\rho_l, \rho_l u_l, \rho_l E_l)$, etc.

It is observed that if $U(x, t)$ is a solution of (3.1) and (3.2) satisfying condition (E), then for any $\alpha > 0$, $U(\alpha x, \alpha t)$ is also a solution satisfying condition (E). We therefore seek solutions of (3.1) and (3.2) which are functions of x/t. We first note that a smooth solution which is a function of x/t is a *centered rarefaction wave* (cf. [3]). This is so, because if $U = U(\delta)$, $\delta = x/t$, is a solution of (3.1), then from (3.1), $U'(\delta) + \delta M U'(\delta) = 0$. Here $U'(\delta) = dU/d\delta$ and M is defined in Section 1. This implies that $U'(\delta)$ is a right eigenvector and δ is an eigenvalue of M. That is, $U(\delta)$ is a centered rarefaction wave.

In what follows, a *centered forward* (*backward*) *wave* is a piecewise smooth solution $U(x, t)$ of (3.1) such that U is a function of x/t and $x/t \geqslant u(x, t)$

$(x/t < u(x, t))$ and such that condition (E) is satisfied across any discontinuity. From above discussions, we know that centered forward (backward) wave consists of centered rarefaction waves and centered shocks associated with R_3 and S_3 (R_1 and S_1) curves. Here R_i are integral curves of r_i, $i = 1, 3$.

THEOREM 3.1. *Given any fixed U_0, there exist smooth curves $\alpha(U_0)$ and $\beta(U_0)$ such that*

(i) *$U_1 \in \alpha(U_0)$ ($\in \beta(U_0)$) if and only if U_1 can be connected to U_0 from the left (right) in a unique way by centered backward (forward) waves.*

(ii) *u and p both are monotonic along $\alpha(U_0)$ and $\beta(U_0)$, and if $\alpha(U_0)$ and $\beta(U_0)$ are so directed that u is increasing along $\alpha(U_0)$ and $\beta(U_0)$, then p is decreasing (increasing) along $\alpha(U_0)$ ($\beta(U_0)$).*

Proof. For the proof of (i), we refer to [4]. To prove (ii), we note that by the construction of α and β, $\alpha(U_0)$ is tangent to R_1 and S_1 curves and $\beta(U_0)$ is tangent to R_3 and S_3 curves. Part (ii) is therefore the consequence of (2.4) and Lemma 2.1. Q.E.D.

THEOREM 3.2. *The Riemann problem* (3.1) *and* (3.2) *has at most one weak solution $U(x, t)$ in the class of centered waves.*

Proof. Suppose that the Riemann problem (3.1) and (3.2) is solved by connecting U_l to U_1 by backward centered waves. U_1 to U_2 by a contact discontinuity, and U_2 to U_r by forward centered waves. Then $U_1 \in \alpha(U_l)$, $U_2 \in \beta(U_r)$ and $u(U_1) = u(U_2) \equiv u^*$, $p(U_1) = p(U_2) \equiv p^*$. Therefore the projections of $\alpha(U_l)$ and $\beta(U_r)$ on the u–p plane meet at (u^*, p^*). On the other hand, (ii) of Theorem 3.1 implies that (u^*, p^*) is uniquely determined by U_l and U_r. Therefore U_1 and U_2 are uniquely determined by U_l and U_r. To complete the proof of the theorem, we note that the backward waves connecting U_l to U_r and forward waves connecting U_2 to U_r have unique forms as asserted in (i) of Theorem 3.1. This completes the proof of the theorem. Q.E.D.

To prove the existence theorem we further assume that $p \to 0_+$ if and only if $\rho \to 0_+$. For simplicity, we also assume that $p_{\tau\tau}$ has only a finite number of zeros, so that $\alpha(U_0)$ and $\beta(U_0)$ consist of R curves or S curves eventually.

THEOREM 3.3. *The Riemann problem* (3.1) *and* (3.2) *has a solution $U(x, t)$ in the class of centered waves such that $u(x, t)$ is always finite, and $p(x, t)$ and $\rho(x, t)$ are positive except possibly for $\alpha_1 < x/t < \alpha_2$; $-\infty < \alpha_1 < \alpha_2 < +\infty$, where p and ρ are zero.*

Proof. Let $\tilde{\alpha}(U_l)$ and $\tilde{\beta}(U_r)$ be the projections of $\alpha(U_l)$ and $\beta(U_r)$ on the u–p plane, respectively. Suppose that $\tilde{\alpha}(U_l)$ and $\tilde{\beta}(U_r)$ intercept. Then it is clear from the proof of Theorem 3.2 that the Riemann problem has a solution $U(x, t)$ such that $u(x, t)$ is finite and $p(x, t)$ and $\rho(x, t)$ are positive.

If $\tilde{\alpha}(U_l)$ and $\tilde{\beta}(U_r)$ do not intercept, then (ii) of Theorem 3.1 implies that one of the following cases must occur.

(i) There exist $\bar{p} > \bar{\bar{p}}$ such that $p \to \bar{p}$ along $\alpha(U_l)$ and $p \to \bar{\bar{p}}$ along $\beta(U_r)$ as $u \to +\infty$,

(ii) there exist $\bar{p} > \bar{\bar{p}}$ such that $p \to \bar{\bar{p}}$ along $\alpha(U_l)$ and $p \to \bar{p}$ along $\beta(U_r)$ as $u \to -\infty$,

(iii) there exist $\bar{u} > \bar{\bar{u}}$ such that $u \to \bar{u}$ along $\alpha(U_l)$ and $u \to \bar{\bar{u}}$ along $\beta(U_r)$ as $p \to +\infty$,

(iv) there exist $\bar{u} > \bar{\bar{u}}$ such that $u \to \bar{\bar{u}}$ along $\alpha(U_l)$ and $u \to \bar{u}$ along $\beta(U_r)$ as $p(\text{or } \rho) \to 0_+$.

Suppose that (iv) holds. Then U_l can be connected to some point U_1 on $\alpha(U_l)$ by backward waves and some point U_2 on $\beta(U_r)$ is connected to U_r by forward waves. Furthermore $u(U_1) = \bar{\bar{u}} < \bar{u} = u(U_2)$ and $p(U_1) = p(U_2) = \rho(U_1) = \rho(U_2) = 0$. Therefore it is obvious that we can connect U_1 to U_2 by setting $p = \rho = 0$ and u an arbitrary finite number. We note that the backward and forward waves do not overlap, because $\bar{\bar{u}} < \bar{u}$. The theorem holds in this case.

It remains to show that cases (i)–(iii) do not hold. We show this only when $\alpha(U_l)$ and $\beta(U_r)$ are composed of rarefaction curves eventually. The case where $\alpha(U_l)$ or $\beta(U_r)$ is tangent to shock curves S eventually can be treated similarly by using the Hugoniot condition.

When (i) holds, then $\lambda_3(U)$ is decreasing as U moves along $\beta(U_r)$ and $u \to +\infty$ (cf. [4]). In particular $\lambda_3(U)$ is bounded from above. On the other hand, $\lambda_3(U) = u + ((-p_2)^{1/2}/\rho) > u$ and so $\lambda_3(U) \to +\infty$ which is a contradiction. Therefore (i) does not hold. Similarly, (ii) does not hold.

When (iii) holds, then $\lambda_1(U)$ is increasing along $\alpha(U_l)$ as $p \to +\infty$ and so $p_{\tau\tau}(U) > 0$ along $\alpha(U_l)$ eventually. For $U \in \alpha(U_l)$, we write $p = p(u)$. It is shown that $d^2p/du^2 < 0$ because $p_{\tau\tau}(U) > 0$. That is, $\tilde{\alpha}(U_l)$ is concave downward. Therefore (iii) cannot be valid.

The proof of the theorem is complete. Q.E.D.

We remark that (3.1) is in Eulerian coordinates; in the Lagrangian coordinates the vacuum region (where $\rho = 0$) shrinks to a single ray. For isentropic gas flow, the occurrence of the state of vacuum along a single ray was observed by Smoller [5].

4. Shock Interaction

The projections of shock and rarefaction curves on the u–p plane are functions of u and also functions of p as asserted in Lemma 2.1. This observation makes it easy to study the shock interaction. In this section, we investigate only the intersection of weak shocks.

THEOREM 4.1. *Suppose that $p_{\tau\tau} > 0$, $p_\tau < 0$ and $\{U_l, U_m\}$ and $\{U_m, U_r\}$ are both forward (backward) shocks satisfying condition* (E). *Then for $|U_l - U_m| + |U_m - U_r|$ sufficiently small, the Riemann problem* (3.1) *and* (3.2) *has a solution consisting of a contact discontinuity and a forward (backward) shock and* (i) *a backward (forward) rarefaction wave if $p_{\tau\tau} > -4p_\tau p_e$, or* (ii) *a backward (forward) shock if $p_{\tau\tau} < -4p_\tau p_e$.*

Proof. We only prove the theorem when $\{U_l, U_m\}$ and $\{U_m, U_r\}$ are both backward shocks. The other case is dealt with similarly. It is known and can be shown easily by (2.4), that if $\{U_1, U_2\}$ is a backward (forward) shock satisfying condition (E) and $p_{\tau\tau} > 0$, then $u(U_1) < u(U_2)$ and $p(U_1) > p(U_2)$ ($p(U_1) < p(U_2)$). Let $U^* \in S_1(U_l)$ with $u(U^*) = u(U_r)$. The proof of the theorem will be complete if we can show that for $|U_l - U_m| + |U_m - U_r|$ small, $p(U^*) < p(U_r)$ if $p_{\tau\tau} > -4p_\tau p_e$, and $p(U^*) > p(U_r)$ if $p_{\tau\tau} < -4p_\tau p_e$.

We denote by d/du the derivative with respect to u along $R_1(U_l)$ and $\partial/\partial u$ the derivative with respect to u along $S_1(U_l)$. Since $R_1(U_m)$ is tangent to $S_1(U_m)$, $U_r \in S_1(U_m)$, and R_1 is a vector field, we have only to show that at $U = U_l$,

$$\text{(i)}\quad \frac{dp}{du} - \frac{\partial p}{\partial u} = \frac{\partial}{\partial u}\left(\frac{dp}{du} - \frac{\partial p}{\partial u}\right) = 0,$$

and

$$\text{(ii)}\quad \frac{\partial^2}{\partial u^2}\left(\frac{dp}{du} - \frac{\partial p}{\partial u}\right) > 0 \text{ if } p_{\tau\tau} < -4p_\tau p_e, \text{ and } \frac{\partial^2}{\partial u^2}\left(\frac{dp}{du} - \frac{\partial p}{\partial u}\right) < 0$$

if $p_{\tau\tau} > -4p_\tau p_e$.

Abbreviating $p_l(\sigma - u_l)$ by ξ, the Hugoniot condition for $\{U_l, U\}$ becomes

$$\xi = \frac{p - p_l}{u - u_l} = \frac{-(u - u_l)}{\tau - \tau_l} = \frac{pu - p_l u_l}{E - E_l}, \tag{4.1}$$

where $p_l = p(U_l)$, $\tau_l = \tau(U_l)$, etc. Differentiating $\xi(u - u_l) = p - p_l$ along $S_1(U_l)$, we have

$$(u - u_l)\frac{\partial \xi}{\partial u} + \xi = \frac{\partial p}{\partial u} \qquad \text{and} \qquad (u - u_l)\frac{\partial^2 \xi}{\partial u^2} + 2\frac{\partial \xi}{\partial u} = \frac{\partial^2 p}{\partial u^2}.$$

Therefore at $U = U_l$,

$$\partial p/\partial u = \xi \qquad \text{and} \qquad \partial^2 p/\partial u^2 = 2(\partial \xi/\partial u). \tag{4.2}$$

On the other hand, differentiating (4.1) along S (U_l) and using equalities $\partial E/\partial u = (\partial e/\partial u) + u$ and $\partial p/\partial u = p_e(\partial e/\partial u) + \tilde{p}_\tau(\partial \tau/\partial u)$, we have for $U \in S_1(U_l)$

$$\frac{\partial p}{\partial u} = \frac{2\xi p_\tau + \xi p_e(p - p_l)}{p_\tau - \xi^2 + p_e(p - p_l)}. \tag{4.3}$$

Evaluating (4.3) at $U = U_l$, we find that

$$\partial p/\partial u = dp/du = \xi = \lambda_1(U_l) \tag{4.4}$$

which is the first equality in (i).

We now differentiate (4.3) once along $S_1(U_l)$ and use (2.4), (4.2), and (4.4) to show that for $U = U_l$

$$\frac{\partial^2 p}{\partial u^2} = \frac{\partial p_\tau/\partial u}{-2\xi} = \frac{p_{\tau\tau}}{2\lambda_1{}^2} = \frac{\partial}{\partial u}\left(\frac{dp}{du}\right). \tag{4.5}$$

This proves (i).

Differentiating (4.3) twice along $S_1(U_l)$ and using (4.2) and (4.4), it follows that at $U = U_l$,

$$\frac{\partial^3 p}{\partial u^3} = \left(\frac{\partial^3 p}{\partial u^2}\frac{\partial p}{\partial u}\left(\frac{5}{2}\frac{\partial^2 p}{\partial u^2} - p_e\right) + \xi\frac{\partial^2 p_\tau}{\partial u^2}\right) - (-\tfrac{1}{2}\xi^{-2}). \tag{4.6}$$

From (2.4), we readily find that for $U = U_l$

$$\frac{\partial^2}{\partial u^2}\left(\frac{dp}{du}\right) = \frac{\lambda_1(d^2p/du^2) + 2(d^2p/du^2)}{-2\lambda_1{}^2}. \tag{4.7}$$

By the same arguments leading to (4.5), we can prove that $d^2p_\tau/du^2 = \partial^2 p_\tau/\partial u^2$. Therefore (4.2) and (4.4)–(4.7) imply that at $U = U_l$

$$\frac{\partial^2}{\partial u^2}\left(\frac{\partial p}{\partial u} - \frac{dp}{du}\right) = \frac{p_{\tau\tau}(p_{\tau\tau} + 4p_\tau p_e)}{8(-\lambda_1)^5}. \tag{4.8}$$

This proves (ii) and hence the theorem. Q.E.D.

EXAMPLE. The constitutive equation for the polytropic gas is (cf. [2])

$$p(\tau, s) = \text{const} \cdot e^{s/c_v}\tau^{-\gamma}, \qquad 1 < \gamma < 2,$$

with

$$p\tau = RT, \quad c_v = R/(\gamma - 1) > 0.$$

Conditions (2.2) are satisfied:

$$\begin{aligned} p_\tau &= -\gamma p/\tau < 0, & p_{\tau\tau} &= \gamma(\gamma + 1)(p/\tau^2) > 0, \\ p_s &= (1/c_v)p > 0, & p_e &= p_e/T = (\gamma - 1)/\tau > 0, \\ \tilde{p}_\tau &= p_\tau + pp_e = -p/\tau < 0. \end{aligned}$$

Therefore, we have $p_{\tau\tau} + 4p_\tau p_e = (p/\tau^2)(\gamma(\gamma + 1) - 4\gamma(\gamma - 1)) = (p\gamma/\tau)(5 - 3\gamma)$. Theorem 4.1 implies the following known result (cf. [1]).

> If $\gamma < 5/3$, then the interaction of two weak forward (backward) shocks produces a forward (backward) shock and a backward (forward) rarefaction wave, if $\gamma > 5/3$, then the interaction of two weak forward (backward) shocks produces forward and backward shocks.

It is also known that when $\gamma = 5/3$, the interaction of two weak forward (backward) shocks produces a backward (forward) rarefaction wave. To prove this, we have to compute the higher derivatives of p along shock and rarefaction curves.

References

1. R. Courant and K. O. Friedricks, National Defense Research Committee, Applied Mathematics Panel Report 38.1 R (1943).
2. R. Courant and K. O. Friedricks, "Supersonic Flow and Shock Waves," Interscience, New York, 1948.
3. P. D. Lax, Hyperbolic systems of conservation laws, II, *Comm. Pure Appl. Math.* **10** (1957), 537–566.
4. T. P. Liu, The Riemann problem for general systems of conservation laws, *J. Differential Equations*, to appear.
5. J. A. Smoller, On the solution of the Riemann problem with general step data for an extended class of hyperbolic systems, *Michigan Math. J.* **16** (1969), 201–210.
6. A. I. Vol'pert, The space BV and quasilinear equations, *Math. USSR-Sb.* **2** (1967), 225–267.
7. B. Wendroff, The Riemann problem for materials with non convex equations of state. II. General flow, *J. Math. Anal. Appl.* **38** (1972), 640–658.